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Course -	**Real Analysis**
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Unit-	**1 (Part-1)**
Topic-	**Reimann-Stieltjes integral**
Sub-Topic-	**Definition and existence of Reimann-Stieltjes integral. Properties of the integral, Integration and differentiation, the fundamental theorem of calculus, Integration of vector-valued functions.**
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UNIT-I

Reimann-Stieltjes integral

Introduction

This short note gives an introduction to the Riemann-Stieltjes integral on R and Rn. Some natural and important applications in probability theory are discussed. The reason for discussing the Riemann-Stieltjes integral instead of the more general Lebesgue and Lebesgue-Stieltjes integrals are that most applications in elementary probability theory are satisfactorily covered by the Riemann-Stieltjes integral. In particular there is no need for invoking the standard machinery of monotone convergence and dominated convergence that hold for the Lebesgue integrals but typically do not for the Riemann integrals.

The reason for introducing Stieltjes integrals is to get a more unified approach to the theory of random variables, in particular for the expectation operator, as opposed to treating discrete and continuous random variables separately. Also it makes it possible to treat mixtures of discrete and continuous random variables: It is for instance not possible to show that the expectation of the sum of a discrete and a continuous r.v. is the sum of the expectations, without using Stieltjes integrals. There are also many advantages in inference theory, for instance in the discussion of plug-in estimators.

The Riemann-Stieltjes integral on R

The Reimann integral corresponds to making no transformation of the x-axis in the Reimann sum

$$\sum_{i=1}^{n} g(\xi_i)(x_{i-1} - x_i).$$

Sometimes one would like to make such a transformation.

Thus let F be a monotone real-valued transformation of I is a subset of R, so

$$F : I \mapsto F(I).$$

Now assume that h is a real-valued step-function on the interval I, so that we can write

$$h(t) \quad = \quad \sum_{i=1}^{n} c_i 1\{t \in I_i\},$$

for some constants c_1, ……, c_n, and with

$$I = \cup_{i=1}^{n} I_i$$

a partition, and with Ij intervals. Then we define the Reimann-Stieltjes integral of h as

$$\int h(u)\, dF(u) \quad = \quad \sum_{i=1}^{n} c_i |F(I_i)|.$$

Note that

$$|F(I)| = F(\max(I_i)) - F(\min(I_i)),$$

and that if

$$I_i = (a_i, b_i) \text{ then } |F(I_i)| = F(b_i) - \ F(b_i),$$

and that then

$$\int_I h(u)\, dF(u) \quad = \quad \sum_{i=1}^{n} c_i (F(b_i) - F(a_i)).$$

We can next make the following definition:

Definition 1: Let F be an increasing function defined on the interval I, and let g be a function defined on I. Then g is called Riemann-Stieltjes integrable, w.r.t. F, if for every $\epsilon > 0$, there are step-functions h1; h2 such that h1 $\leq$ g $\leq$ h2 and

$$\int_I h_2(u)\, dF(u) - \int_I h_1(u)\, dF(u) \quad < \quad \epsilon.$$

If g is Riemann-Stieltjes integrable, we define the Riemann-Stieltjes integral of g as

$$\int_I g(u)\, dF(u) \quad = \quad \sup\{\int_I h(u)\, dF(u) : h \leq g, h \textit{ step function.}\}$$

Theorem 1: If g is continuous and bounded, and F is increasing and bounded on the interval I, then g is Riemann-Stieltjes integrable on I.

Proof. That F is increasing and bounded on I = [a, b] means that

$$-\infty < c := F(a) = \inf_I F \leq \sup_I F = F(b) =: C < \infty.$$

(i): Assume first that I is finite. Thus since g is continuous on the compact interval I it is uniformly continuous so there are ϵ, δ such that

$$|x - y| \leq \delta \quad \Rightarrow \quad |g(y) - g(x)| \leq \epsilon.$$

$$\cup_{i=1}^n I_i = I$$

for ϵ, δ not $|F(I_i)| \leq \delta$ depending on x, y. Next let be an arbitrary finite partition of I, with I_i intervals that satisfy (this is possible to obtain since F is bounded), and let

$$\begin{aligned} m_i &= \inf_{t \in I_i} g(t), \\ M_i &= \sup_{t \in I_i} g(t). \end{aligned}$$

and $M_i - m_i < \epsilon$, note that by the uniform continuity of g. The step functions

$$
\begin{aligned}
h_1(t) &= \sum_{i=1}^{n} m_i 1\{t \in I_i\}, \\
h_2(t) &= \sum_{i=1}^{n} M_i 1\{t \in I_i\},
\end{aligned}
$$

$h_1 \leq g \leq h_2$. satisfy Furthermore

$$
\int_I h_1(u)\, dF(u) = \sum_{i=1}^{n} m_i |F(I_i)| \;\; \leq \;\; \sum_{i=1}^{n} M_i |F(I_i)| = \int_I h_2(u)\, dF(u),
$$

so that

$$
\int_I h_2(u)\, dF(u) - \int_I h_1(u)\, dF(u) \;\; \leq \;\; \sum_{i=1}^{n} (M_i - m_i)|F(I_i)| \leq \epsilon \sum_{i=1}^{n} |F(I_i)| = \epsilon |F(I)|,
$$

where the last equality follows since the sets $F(I_i)$ are disjoint (by the monotonicity of F together with the fact that I_i are disjoint). Since for every $\in > 0$ we can get h1, h2 step functions so that this holds, we have shown that g is Riemann-Stieltjes integrable.

(ii): Next, assume I not finite. Then since F is increasing it is also piecewise continuous. Therefore for every $\tilde{\in} > 0$ there is finite $\tilde{I} \in I$ such that

$$
\max(\sup_I F - \sup_{\tilde{I}} F, \inf_{\tilde{I}} F - \inf_I F) \;\; < \;\; \tilde{\epsilon}.
$$

Also, since g is absolutely bounded,

$$
\sup_{I \setminus \tilde{I}} |g| \leq G
$$

so that

$$
\begin{aligned}
&-G \leq g \leq G \text{ on } I \setminus \tilde{I}, \\
\int_{I \setminus \tilde{I}} G\, dF(u) - \int_{I \setminus \tilde{I}} -G\, dF(u) \;\; &\leq \;\; 2G\tilde{\epsilon}.
\end{aligned}
$$

Thus we can use the construction under (i) on $\tilde{I}$, and concatenate to get the step functions

$$\tilde{h}_1 = \mathrm{conc}(-G, h_1, -G), \tilde{h}_2 = \mathrm{conc}(G, h_2, G)$$

bounding g on all of I and such that

$$\begin{aligned}\int_I \tilde{h}_2\, dF(u) - \int_I \tilde{h}_1\, dF(u) &= \int_{\tilde{I}} h_2(u)\, dF(u) - \int_{\tilde{I}} h_1(u)\, dF(u) \\ &\quad + \int_{I\setminus\tilde{I}} G\, dF(u) - \int_{I\setminus\tilde{I}} -G\, dF(u) \\ &\leq \epsilon|F(\tilde{I})| + 2G\tilde{\epsilon}.\end{aligned}$$

Since $\in, \tilde{\in}$ are arbitrary we have shown that g is Riemann-Stieltjes integrable.

The Riemann-Stieltjes integral can be obtained as a limit of Riemann-Stieltjes sums. We prove the statements for continuous functions g:

Theorem 2. Assume g is continuous and F increasing on the interval I. Then

$$\int_I g(x)\, dFx = \lim_{\max_{1\leq i\leq n}|x_i - x_{i-1}|\to 0} \sum_{i=1}^{n} g(\xi_i)(F(x_i) - F(x_{i-1})),$$

$\min I = x_0 < x_1 < \ldots < x_n < \max I$ where are partitions of I, and ξ are arbitrary points in $[x_{i-1}, x_i)$.

Proof. Use a similar construction of h1; h2 as in the proof of Theorem 1. Thus we have

$$h_1 \leq g \leq h_2.$$

and

$$\int_I h_1(u)\, dF(u) \leq \sum_{i=1}^{n} g(\xi_i)(F(x_i) - F(x_{i-1})) \leq \int_I h_2(u)\, dFu.$$

Since g is integrable, we $\epsilon \downarrow 0,$ can let and make the partition $n \to \infty,$ finer and finer as

so that the difference between the right hand side and the left hand side (which is smaller than $\in$) goes to zero, which shows the result.

The Integral

The area of a circle: Egyptian knew how to calculate before 1650 B. C.

The general method for calculating the area: Archimedes (287 B. C.~212 B. C.) proposed the method of exhaustion.

Antiderivatives

Definition 1:

F is an antiderivative of f if $F'(x) = f(x)$.

Example 1:

$f(x) = 3x^2 \Rightarrow F(x) = x^3 + c$, where ***c*** is a constant.

◆

Theorem 1:

If F and G are two differentiable functions that have the same derivative in (a,b), i.e., $F'(x) = G'(x)$. Then, $F(x) - G(x) = c$, where ***c*** is any constant.

Definition 2:

Let F be an antiderivative for f, then the indefinite integral of f is written $\int f(x)dx = F(x) + c$, where f is referred to as the integrand, x is referred to as the variable of integration and ***c*** is any constant. If f has an antiderivative, then f is said to be ***integrable***.

Theorem 2:

$$\int k\,dx = kx + c,\ \int x^r\,dx = \frac{x^{r+1}}{r+1} + c,$$
$$\int \sin(x)\,dx = -\cos(x) + c,\ \int \cos(x)\,dx = \sin(x) + c$$

Theorem 3:

1. $\int kf(x)\,dx = k\int f(x)\,dx$.

2. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.

[justifications:]

1. Let $\int f(x)\,dx = F$. Since

$$\frac{d(kF)}{dx} = \frac{kdF}{dx} = kf(x),$$

$$kF = k\int f(x)\,dx = \int kf(x)\,dx$$

by theorem 1.

2. Let $\int f(x)\,dx = F$ and $\int g(x)\,dx = G$. Since

$$\frac{d(F+G)}{dx} = \frac{dF}{dx} + \frac{dG}{dx} = f(x) + g(x),$$

$$F + G = \int f(x)\,dx + \int g(x)\,dx = \int [f(x) + g(x)]\,dx$$

by theorem 1.

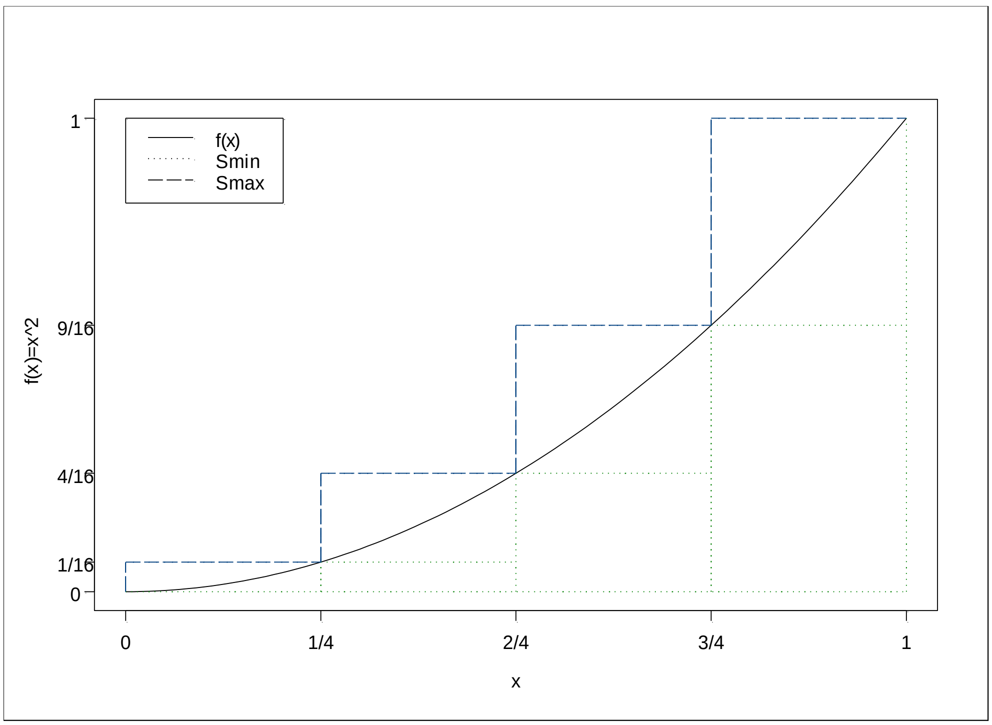
Theorem 4:

If f is continuous, then f is ***integrable***.

Properties of the integral

The Definite Integral

Motivating Example :



In the above figure, the area, ***A***, between $f(x)=x^2$ over $[0,1]$ and the **x-axis** can be approximated by the rectangles in dashed lines or dotted lines. The area of the rectangles in dotted lines is

$$
\begin{aligned}
S_{\min} &= \left(\frac{1}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{2}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} \\
&= \frac{1}{16} \cdot \frac{1}{4} + \frac{4}{16} \cdot \frac{1}{4} + \frac{9}{16} \cdot \frac{1}{4} \\
&= \frac{7}{32}
\end{aligned}
$$

while the area of the rectangles in dashed lines is

$$
\begin{aligned}
S_{\max} &= \left(\frac{1}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{2}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{4}{4}\right)^2 \cdot \frac{1}{4} \\
&= \frac{1}{16} \cdot \frac{1}{4} + \frac{4}{16} \cdot \frac{1}{4} + \frac{9}{16} \cdot \frac{1}{4} + \frac{16}{16} \cdot \frac{1}{4} \\
&= \frac{15}{32}
\end{aligned}
$$

Thus,

$$0.22 \approx \frac{7}{32} < A < \frac{15}{32} \approx 0.47 .$$

In the above approximation, the interval $[0,1]$ is divided into subintervals of equal length,

$$\left[0, \tfrac{1}{4}\right], \left[\tfrac{1}{4}, \tfrac{2}{4}\right], \left[\tfrac{2}{4}, \tfrac{3}{4}\right], \left[\tfrac{3}{4}, 1\right].$$

If the interval $[0,1]$ is divided into more subintervals of equal lengths, for example,

$$\left[0,\tfrac{1}{8}\right],\left[\tfrac{1}{8},\tfrac{2}{8}\right],\left[\tfrac{2}{8},\tfrac{3}{8}\right],\left[\tfrac{3}{8},\tfrac{4}{8}\right],\left[\tfrac{4}{8},\tfrac{5}{8}\right],\left[\tfrac{5}{8},\tfrac{6}{8}\right],\left[\tfrac{6}{8},\tfrac{7}{8}\right],\left[\tfrac{7}{8},1\right]$$

then the area ***A*** can be approximated by similar rectangles in dashed lines or dotted lines. The area of the rectangles in dotted lines is

$$\begin{aligned} S_{\min} &= \left(\frac{1}{8}\right)^2 \cdot \frac{1}{8} + \left(\frac{2}{8}\right)^2 \cdot \frac{1}{8} + \cdots + \left(\frac{7}{8}\right)^2 \cdot \frac{1}{8} \\ &= \frac{35}{128} \end{aligned}$$

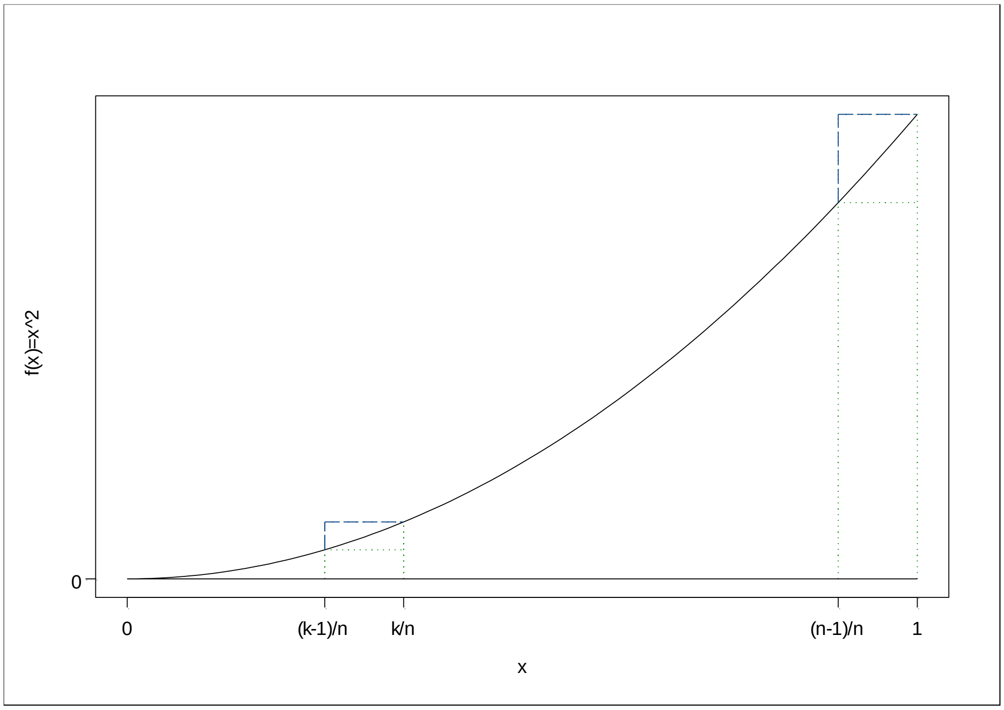
while the area of the rectangles in dashed lines is

$$\begin{aligned} S_{\max} &= \left(\frac{1}{8}\right)^2 \cdot \frac{1}{8} + \left(\frac{2}{8}\right)^2 \cdot \frac{1}{8} + \cdots + \left(\frac{7}{8}\right)^2 \cdot \frac{1}{8} + \left(\frac{8}{8}\right)^2 \cdot \frac{1}{8} \\ &= \frac{51}{128} \end{aligned}$$

Thus,

$$0.27 \approx \frac{35}{128} < A < \frac{51}{128} \approx 0.40\,.$$

A more accurate approximation can be obtained. In general, the interval $[0,1]$ is divided into subintervals with equal length $\tfrac{1}{n}$.



The area of the rectangles in dotted lines is

$$S_{\min} = \left(\frac{1}{n}\right)^2 \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^2 \cdot \frac{1}{n} + \cdots + \left(\frac{n-1}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \frac{1^2 + 2^2 + \cdots + (n-1)^2}{n^3}$$

$$= \frac{1}{n^3} \sum_{k=1}^{n-1} k^2$$

$$= \frac{(n-1)n(2n-1)}{6n^3}$$

while the area of the rectangles in dashed lines is

$$
\begin{aligned}
S_{\min} &= \left(\frac{1}{n}\right)^2 \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^2 \cdot \frac{1}{n} + \cdots + \left(\frac{n-1}{n}\right)^2 \cdot \frac{1}{n} + \left(\frac{n}{n}\right)^2 \cdot \frac{1}{n} \\
&= \frac{1^2 + 2^2 + \cdots + (n-1)^2 + n^2}{n^3} \\
&= \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\
&= \frac{n(n+1)(2n+1)}{n^3}
\end{aligned}
$$

Thus,

$$\frac{(n-1)n(2n-1)}{6n^3} < A < \frac{n(n+1)(2n+1)}{6n^3}$$

As ***n*** tends to infinity, by squeezing theorem,

$$\lim_{n\to\infty} \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3} = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3},$$

$$A = \lim_{n\to\infty} A = \frac{1}{3}$$

Note:

In the above approximations, the same result can be obtained as the heights of the rectangles are replaced by $f\left(x_i^*\right)$, where

$$x_i^* = \frac{(i-1)}{n} + c,\ 0 < c < \frac{1}{n},\ i = 1, 2, \ldots, n.$$

That is, rather than using $f\left(\frac{i}{n}\right)$ or $f\left(\frac{(i-1)}{n}\right)$, the values of the function evaluating at the inner points of the subintervals are used as the heights of the rectangles. For example, as using the middle points of these subintervals as the heights,

$$x_i^* = \frac{(i-1)}{n} + \frac{1}{2n} = \frac{2i-1}{2n}, i = 1, 2, \ldots, n,$$

$$\Rightarrow \text{approximated area} = \left(\frac{1}{2n}\right)^2 \cdot \frac{1}{n} + \left(\frac{3}{2n}\right)^2 \cdot \frac{1}{n} + \cdots + \left(\frac{2n-1}{2n}\right)^2 \cdot \frac{1}{n}$$

$$= \frac{1^2 + 3^2 + \cdots + (2n-1)^2}{4n^3} = \frac{\sum_{k=1}^{2n-1} k^2 - \sum_{k=1}^{n-1} (2k)^2}{4n^3}$$

$$= \frac{4n^2 - 1}{12n^2}$$

Thus, as *n* tends to infinity, the approximated area tends to $1/3$.

Definition 3 (Riemann sum and regular partition):

Let

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Let $x_i - x_{i-1} = \Delta x_i, i = 1, 2, \ldots, n$, The Riemann sum is

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i,\ x_i^* \in [x_{i-1}, x_i].$$

As

$$\Delta x_i = \frac{b-a}{n}$$

the partition is regular.

Definition 4 (the definite integral):

Let f be defined on $[a,b]$. Then,

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i ,$$

whenever the limit exists.

Note:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n}, \Delta x = \frac{b-a}{n} .$$

Theorem 5:

f is continuous on $[a,b]$, then f is integrable on $[a,b]$. That is,

$\int_a^b f(x)dx$ exists.

Note:

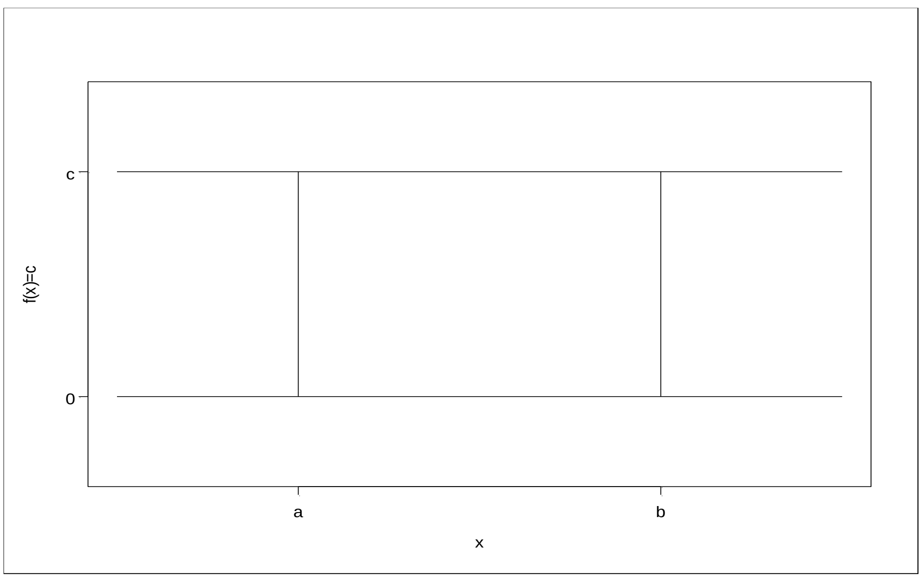
f is continuous, then

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n}, \Delta x = \frac{b-a}{n}$$

Theorem 6:

Let c be a constant. Then,

$$\int_a^b c dx = c(b-a).$$



Definition 5:

For any real number ***a***,

$$\int_a^a f(x)dx = 0.$$

Definition 6:

If $a < b$ and $\int_a^b f(x)dx$ exists, then

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

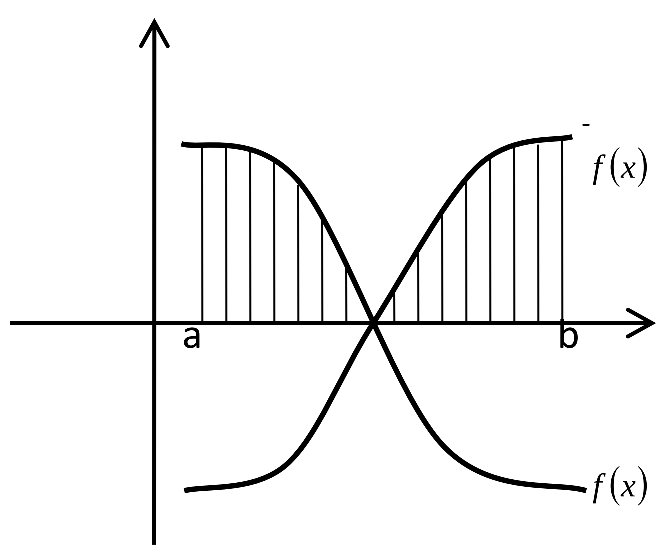
Example 2:

$$\int_1^0 x^2 dx = -\int_0^1 x^2 dx = \frac{-1}{3}.$$

Definition 7 (area):

The area bounded by the function $y = f(x)$, is denoted by A_a^b and is defined by the formula

$$A_a^b = \int_a^a |f(x)| dx.$$



Properties of Definite Integral:

Theorem 7:

If f is continuous on $[a,b]$ and if $a<c<b$, then f is integrable on $[a,c]$ and on $[c,b]$, and

$$\int_b^a f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Theorem 8:

f is integrable on $[a,b]$ and if k is any constant, then kf is integrable on $[a,b]$, and

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx.$$

Theorem 9:

If the function f and g are both integrable on $[a,b]$, then $f+g$ is integrable on $[a,b]$, and

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Theorem 10:

If f is integrable on $[a,b]$ and $f \geq 0$ there, then

$$\int_a^b f(x)dx \geq 0.$$

Theorem 11:

If the function f and g are both integrable on $[a,b]$ and $f(x) \leq g(x)$, then

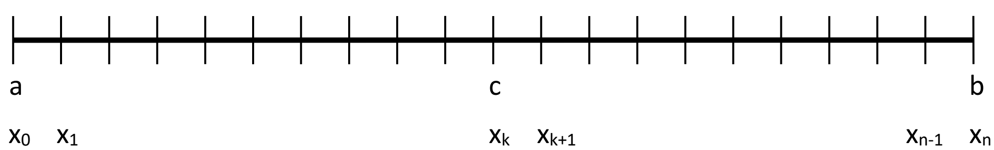
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Theorem 12:

If f is integrable on $[a,b]$ and $m \leq f \leq M$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

[justifications of theorem 7:]



Since

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \sum_{i=1}^{k} f(x_i^*)\Delta x_i + \sum_{i=k+1}^{n} f(x_i^*)\Delta x_i,$$

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\max \Delta x_i \to 0} \left[\sum_{i=1}^{k} f(x_i^*)\Delta x_i + \sum_{i=k+1}^{n} f(x_i^*)\Delta x_i \right]$$

$$= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{k} f(x_i^*)\Delta x_i + \lim_{\max \Delta x_i \to 0} \sum_{i=k+1}^{n} f(x_i^*)\Delta x_i$$

$$= \int_a^c f(x)dx + \int_c^b f(x)dx$$

[justifications of theorem 8:]

$$k\int_a^b f(x)dx = k \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\max \Delta x_i \to 0} k\sum_{i=1}^{k} f(x_i^*)\Delta x_i$$

$$= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{k} kf(x_i^*)\Delta x_i$$

$$= \int_a^b kf(x)dx$$

[justifications of theorem 9:]

Since f and g are both integrable, then

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

and

$$\int_a^b g(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} g(x_i^*)\Delta x_i .$$

$$
\begin{aligned}
\int_a^b f(x)dx + \int_a^b g(x)dx &= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i + \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} g\left(x_i^*\right)\Delta x_i \\
&= \lim_{\max \Delta x_i \to 0} \left[\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i + \sum_{i=1}^{n} g\left(x_i^*\right)\Delta x_i \right] \\
&= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{k} \left[f\left(x_i^*\right) + g\left(x_i^*\right)\right]\Delta x_i \\
&= \int_a^b \left[f(x) + g(x)\right] dx
\end{aligned}
$$

[justifications of theorem 10:]

Since

$$
f\left(x_i^*\right) \geq 0, \Delta x > 0 \;\Rightarrow\; \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i \geq 0, \forall x_i ,
$$

$$
\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i \geq 0.
$$

[justifications of theorem 11:]

Since $h(x) = g(x) - f(x) \geq 0$ and

$$
h(x) = g(x) - f(x) = g(x) + (-1 \cdot f(x))
$$

is integrable (by theorems 8 and 9), then

$$
\int_a^b h(x)dx \geq 0 \,(\text{by theorem } 10).
$$

Thus,

$$0 \le \int_a^b h(x)dx = \int_a^b [g(x) + (-1 \cdot f(x))]dx \overset{\text{by theorem9}}{=} \int_a^b g(x)dx + \int_a^b (-1 \cdot f(x))dx$$

$$\overset{\text{by theorem8}}{=} \int_a^b g(x)dx + (-1) \cdot \int_a^b f(x)dx$$

$$= \int_a^b g(x)dx - \int_a^b f(x)dx$$

[justifications of theorem 12:]

Let $g(x) = M$ and thus $g(x)$ is integrable. Then, $f(x) \le g(x)$. By theorems 11 and 6,

$$\int_a^b f(x)dx \le \int_a^b g(x)dx = \int_a^b Mdx = M(b-a).$$

Similarly,

$$\int_a^b f(x)dx \ge \int_a^b mdx = m(b-a).$$

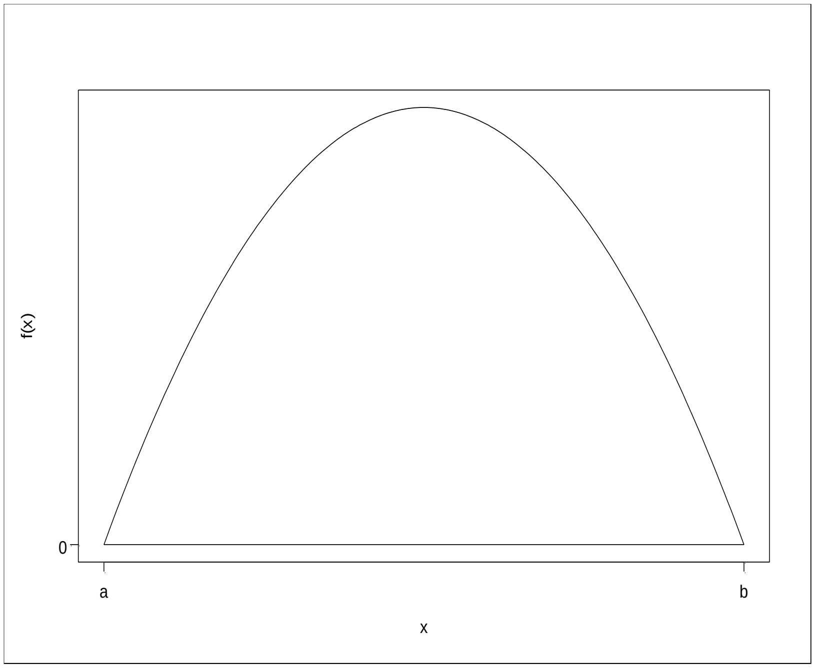
4.3 The Fundamental Theorem of Calculus

Theorem 13 (Rolle's theorem):

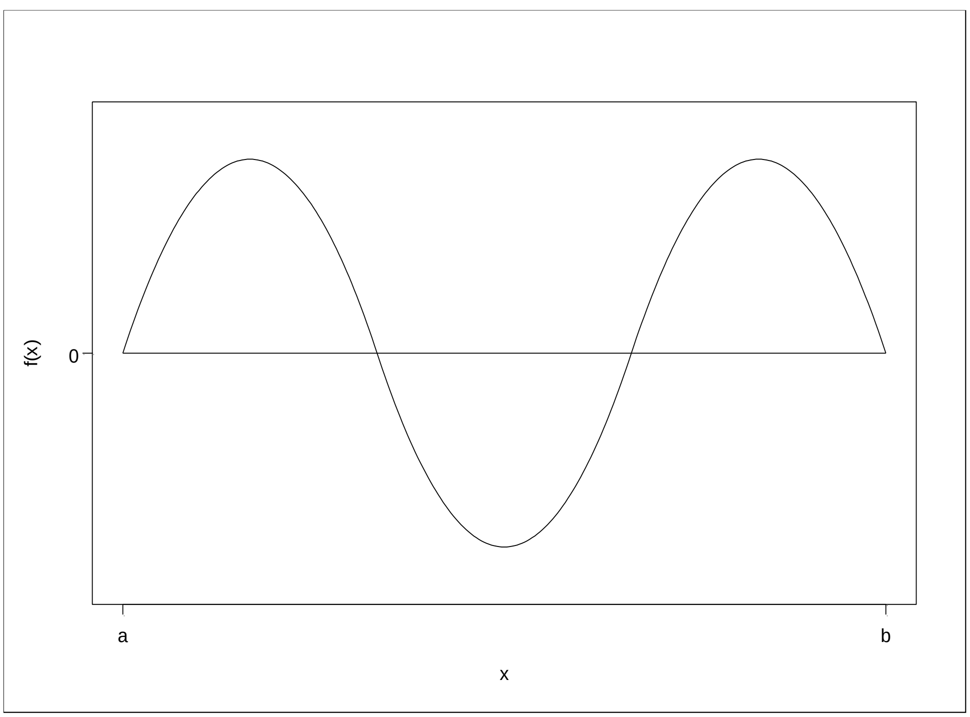
Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b) = 0$, then there exists at least one number ***c*** in (a,b) at which $f'(c) = 0$.

[Intuitions:]

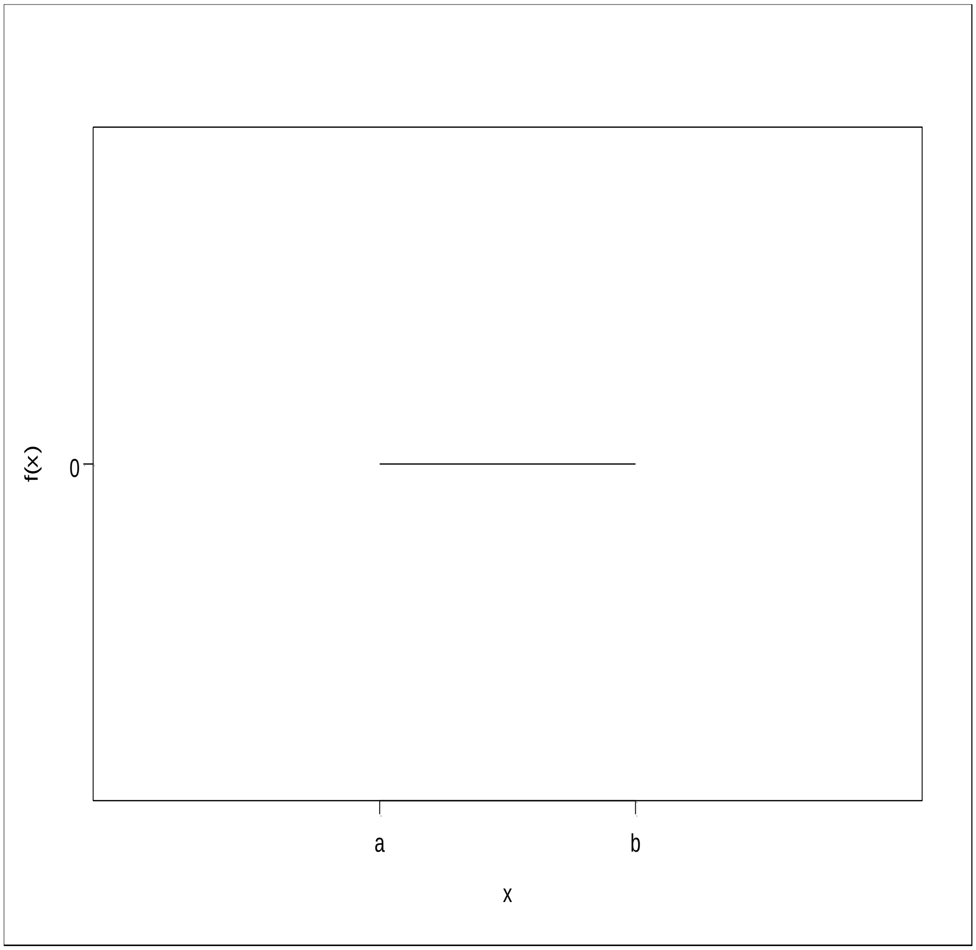
(1)



(2)



(3)



If (3) (figure), $f^{'}(x)=0$ in (a,b). Thus, $f^{'}(c)=0, \forall c \in (a,b)$. If

(1) or (2), suppose f takes on some positive values in (a,b). Intuitive, there is a number x_1 in $[a,b]$, such that $f(x_1)=M>0$, where ***M*** is the maximum value of $f(x)$ in $[a,b]$. Then, $f^{'}(x_1)=0$.

Theorem 14 (mean-value theorem):

Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)=0$, then there exists at least one number ***c*** in (a,b) at which

$$f(b)-f(a)=f'(c)(b-a).$$

[justifications of theorem 14:]

$$h(x)=\left[\frac{f(b)-f(a)}{b-a}\right]\cdot(x-a)+f(a).$$

Then, let $g(x)=f(x)-h(x)$. Since

$$g(a)=f(a)-h(a)=0, g(b)=f(b)-h(b)=f(b)-f(b)=0,$$

by Rolle's theorem, there is a number ***c*** such that

$$g'(c)=0 \Rightarrow g'(c)=f'(c)-\left[\frac{f(b)-f(a)}{b-a}\right]=0$$

$$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}.$$

The fundamental theorem of calculus is the core of calculus. The following example provides the intuition of the theorem.

Motivating Example (continue):

The area, ***A***, bounded by $f(x)=x^2$ over $[0,1]$ is $1/3$. Note that the antiderivative of f is $F(x)=x^3/3+c$ and $F'(x)=f(x)$. As the interval $[0,1]$ is divided into subintervals with equal length $1/n$, the approximated area is

$$\int_0^1 f(x)dx \approx \left(x_1^*\right)^2 \cdot \frac{1}{n}+\left(x_2^*\right)^2 \cdot \frac{1}{n}+\cdots+\left(x_n^*\right)^2 \cdot \frac{1}{n},\ x_i^* \in \left[\frac{i-1}{n},\frac{i}{n}\right].$$

By mean-value theorem,

$$\begin{aligned} F\left(\frac{i}{n}\right)-F\left(\frac{i-1}{n}\right) &= F'(c_i)\cdot\left(\frac{i}{n}-\frac{i-1}{n}\right) \\ &= c_i^2 \cdot \frac{1}{n} \end{aligned}$$

where $c_i \in \left(\frac{i-1}{n},\frac{i}{n}\right)$. As x_i^* is chosen such that $x_i^*=c_i$, the approximated area is

$$\int_0^1 f(x)dx$$
$$\approx (x_1^*)^2 \cdot \frac{1}{n} + (x_2^*)^2 \cdot \frac{1}{n} + \cdots + (x_n^*)^2 \cdot \frac{1}{n},$$
$$= (c_1^*)^2 \cdot \frac{1}{n} + (c_2^*)^2 \cdot \frac{1}{n} + \cdots + (c_n^*)^2 \cdot \frac{1}{n},$$
$$= \left[F\left(\frac{1}{n}\right) - F\left(\frac{0}{n}\right)\right] + \left[F\left(\frac{2}{n}\right) - F\left(\frac{1}{n}\right)\right] + \cdots + \left[F\left(\frac{n-1}{n}\right) - F\left(\frac{n-2}{n}\right)\right] + \left[F\left(\frac{n}{n}\right) - F\left(\frac{n-1}{n}\right)\right]$$
$$= F(1) - F(0) = \frac{1}{3} - \frac{0}{3}$$
$$= \frac{1}{3}$$

Thus, it is nature to ask if in general for a function f with antiderivative F

$$\int_a^b f(x)dx = F(b) - F(a).$$

Theorem 15 (fundamental theorem of calculus):

Let f be continuous on $[a,b]$. If F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

[justifications of theorem 15:]

Since f be continuous on $[a,b]$, then $\int_a^b f(x)dx$ exists. Let

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Then, by mean value theorem,

$$
\begin{aligned}
&F(b)-F(a)=F(x_n)-F(x_0)\\
&=[F(x_1)-F(x_0)]+[F(x_2)-F(x_1)]+\cdots+[F(x_{n-1})-F(x_{n-2})]+[F(x_n)-F(x_{n-1})]\\
&=F'(x_1^*)(x_1-x_0)+F'(x_2^*)(x_2-x_1)+\cdots+F'(x_n^*)(x_n-x_{n-1})\\
&=\sum_{i=1}^{n}F'(x_i^*)\Delta x_i\\
&=\sum_{i=1}^{n}f(x_i^*)\Delta x_i
\end{aligned}
$$

where $\Delta x_i = x_i - x_{i-1}$ and $x_i^* \in (x_{i-1}, x_i)$. Thus,

$$F(b)-F(a)=\lim_{\max \Delta x_i \to 0}[F(b)-F(a)]=\lim_{\max \Delta x_i \to 0}\left[\sum_{i=1}^{n}f(x_i^*)\Delta x_i\right]=\int_a^b f(x)dx .$$

Example 3:

Calculate $\int_1^3 x^3dx$.

[solutions:]

Since the antiderivative of x^3 is $F(x)=\frac{x^4}{4}+c$, by the fundamental theorem of calculus,

$$\int_1^3 x^3dx = F(3)-F(1)=\left[\frac{3^4}{4}+c\right]-\left[\frac{1^4}{4}+c\right]=\frac{81}{4} .$$

Note:

For convenience, the notation,

$$F(x)\Big|_a^b = F(b) - F(a),$$

is used.

Theorem 16 (second fundamental theorem of calculus):

If f be continuous on $[a,b]$, then

$$G(x) = \int_a^x f(t)dt,$$

is continuous, differentiable on (a,b), and for every ***x*** in (a,b),

$$G'(x) = f(x).$$

[Intuition of theorem 16:]

Suppose $f(x)$ is positive. The area bounded by $f(x)$ over $[a,x]$ is

$$A_a^x = \int_a^x f(x)dx = G(x).$$

Then,

$$G'(x) = \lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{A_a^{x+\Delta x} - A_a^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{A_x^{x+\Delta x}}{\Delta x}$$

In the above figure,

$$f(x+\Delta x)\Delta x \le A_x^{x+\Delta x} \le f(x)\Delta x$$

$$\Rightarrow \; f(x+\Delta x) \le \frac{A_x^{x+\Delta x}}{\Delta x} \le f(x)$$

By squeezing theorem, since

$$\lim_{\Delta x\to 0} f(x+\Delta x) = f(x) = \lim_{\Delta x\to 0} f(x),$$

$$G^{'}(x) = \lim_{\Delta x\to 0} \frac{A_x^{x+\Delta x}}{\Delta x} = f(x).$$

4.4 Integration by Substitution and Differentials

Theorem 17:

$$\int f[g(x)]g^{'}(x)dx = F[g(x)] + c,$$

where F is an antiderivative of f and ***c*** is some constant.

[justifications of theorem 17:]

$$\frac{dF[g(x)]}{dx} = F^{'}[g(x)]g^{'}(x) = f[g(x)]g^{'}(x).$$

Example 4:

Calculate $\int (x^2+1)^2 2x dx$.

[solutions:]

Let

$$f(x)=x^2,\ g(x)=x^2+1 \Rightarrow g^{'}(x)=2x,\ F(x)=x^3\big/3,\ f(g(x))=(x^2+1)^2.$$

By theorem 17,

$$\int (x^2+1)^2 2x dx = \int f[g(x)]g^{'}(x)dx = F[g(x)]+c = \frac{(x^2+1)^3}{3}+c.$$

Note:

For the purpose of computations, the following procedure can be used to obtain the integral :

$$u=g(x), g^{'}(x)=\frac{dg(x)}{dx} \Rightarrow du=dg(x)=g^{'}(x)dx$$
$$\Rightarrow \int f[g(x)]g^{'}(x)dx=\int f(u)du=F(u)+c=F[g(x)]+c$$

Example 4 (continue):

Calculate $\int (x^2+1)^2 2x dx$.

[solutions:]

Let

$$u=g(x)=x^2+1 \Rightarrow \frac{dg(x)}{dx}=g^{'}(x)=2x \Rightarrow du=dg(x)=g^{'}(x)dx.$$

$$\int (x^2+1)^2 2x\,dx = \int f[g(x)]g'(x)dx = \int f[g(x)]dg(x) = \int f(u)du$$

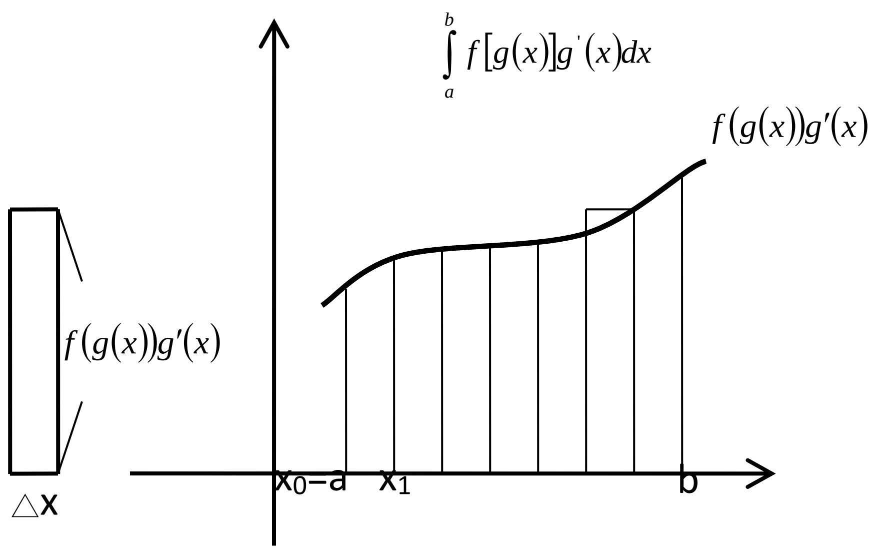
$$= \int u^2 du = \frac{u^3}{3} + c = \frac{(x^2+1)^3}{3} + c$$

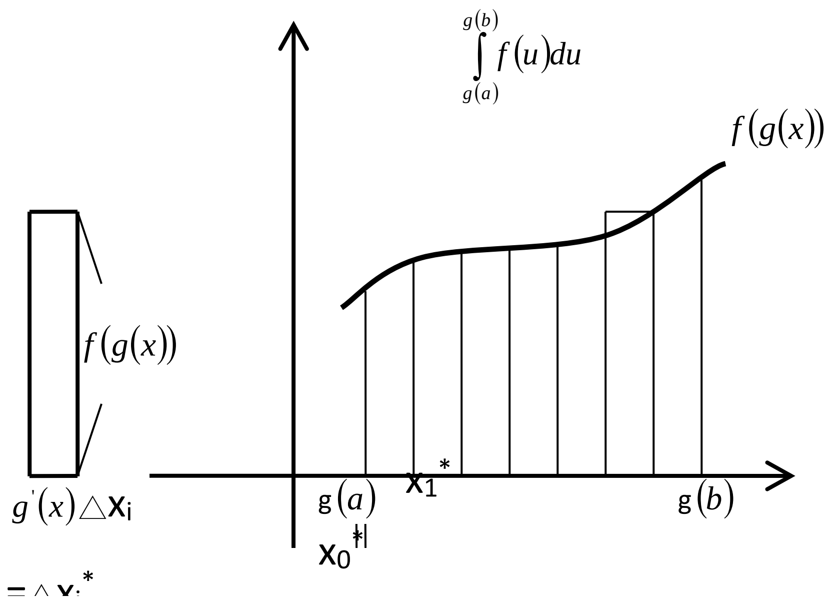
Theorem 18:

If the function $u = g(x)$ has a continuous derivative on $[a,b]$, and f is continuous on the range of $g(x)$,

$$\int_a^b f[g(x)]g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

[Intuition of theorem 18:]





Let

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and

$$g(a) = x_0^* < x_1^* = g(x_1) < x_2^* = g(x_2) < \cdots < x_{n-1}^* = g(x_{n-1}) < x_n^* = g(b).$$

Note that

$$\begin{aligned} \Delta x_i = x_i - x_{i-1} \approx 0 &\Rightarrow \Delta x_i^* = x_i^* - x_{i-1}^* = g(x_i) - g(x_{i-1}) \approx 0 \\ &\Rightarrow \frac{g(x_i) - g(x_{i-1})}{\Delta x_i} \approx g'(x_{i-1}) \approx g'(x_i) \\ &\Rightarrow g(x_i) - g(x_{i-1}) \approx g'(x_i)\Delta x_i \\ &\Rightarrow \Delta x_i^* \approx g'(x_i)\Delta x_i \end{aligned}$$

Then,

$$\int_a^b f[g(x)]g^{'}(x)dx$$
$$\approx f[g(x_1)]g^{'}(x_1)\Delta x_1 + f[g(x_2)]g^{'}(x_2)\Delta x_2 + \cdots + f[g(x_n)]g^{'}(x_n)\Delta x_n$$
$$\approx f[g(x_1)]\Delta x_1^* + f[g(x_2)]\Delta x_2^* + \cdots + f[g(x_n)]\Delta x_n^*$$
$$= f(x_1^*)\Delta x_1^* + f(x_2^*)\Delta x_2^* + \cdots + f(x_n^*)\Delta x_n^*$$
$$\approx \int_{x_0^*}^{x_n^*} f(u)du$$
$$= \int_{g(a)}^{g(b)} f(u)du$$

Example 5:

Calculate $\int_0^1 (x^2+1)^2 2x dx$.

[solutions:]

Let

$$f(x) = x^2, u = g(x) = x^2 + 1 \Rightarrow F(x) = x^3/3.$$

By theorem 18,

$$\int_0^1 (x^2+1)^2 2xdx = \int_0^1 f[g(x)]g^{'}(x)dx = \int_{g(0)}^{g(1)} f(u)du = \int_1^2 u^2 du = \left.\frac{u^3}{3}\right|_1^2$$
$$= \frac{2^3}{3} - \frac{1^3}{3}$$
$$= \frac{7}{3}$$

Note:

For the purpose of computations, the following procedure can be used to obtain the definite integral :

1. The indefinite integral was computed first,

$$u = g(x), g^{'}(x) = \frac{dg(x)}{dx} \Rightarrow du = dg(x) = g^{'}(x)dx$$
$$\Rightarrow \int f[g(x)]g^{'}(x)dx = \int f(u)du = F(u) + c$$

2. Evaluate $F(u)\Big|_{g(a)}^{g(b)} = F[g(b)] - F[g(a)]$.

Example 5 (continue):

1. The indefinite integral is

$$f(x) = x^2, u = g(x) = x^2 + 1$$

$$\int (x^2 + 1)^2 2x dx = \frac{u^3}{3} + c.$$

2.

$$\frac{u^3}{3}\Big|_{g(0)}^{g(1)} = \frac{u^3}{3}\Big|_{0^2+1}^{1^2+1} = \frac{u^3}{3}\Big|_{1}^{2} = \left[\frac{2^3}{3} - \frac{1^3}{2}\right] = \frac{7}{3}.$$

References:

1. Real Analysis, Pragati Publication, Meerut.
2. Real Analysis, Krishna Publication, Meerut.